Common Asset Impact on Default Contagion

Osvaldo Paulo Israel Cançado Assunção

Bradesco Asset Management

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Introduction and Motivation

- Consider a system of banks with interbank loans;
- Banks have investments outside the banking system, for instance stocks and/or real estate;
- When one (or a few) banks default it could create a default cascade in the system;
- We consider the case when banks suffer a sudden capital loss due to a price drop of an asset external to the banking system;
- The goal is to study how common assets can impact default contagion in this system.

Introduction and Motivation

- We consider a network of banks which is given by the interbank loans;
- We analyze the joint effect of loss in a common asset and default contagion .
- 'Ignoring the compounded effect of correlated market shocks and contagion via counterparty exposures can lead to a serious underestimation of contagion risk', [Cont et al., 2013];
- 'Contagion is rare but can nonetheless wipe out a major part of the banking system', [Elsinger et al., 2006];

Related Literature

- Default contagion: [Eisenberg and Noe, 2001] with given graph, extensions *et al.* [Banerjee et al., 2018], and [Banerjee and Feinstein, 2018];
- Random graph approach: [Gai and Kapadia, 2010], *et al.* [Amini et al., 2016], [Detering et al.,] [Detering et al., 2019];
- Empirical studies: [Cont et al., 2013] [Cont and Wagalath, 2014] [Cont and Schaanning, 2014], [Elsinger et al., 2006];
- Common assets: [Allen et al., 2012], [Braverman and Minca, 2018], [Kley and Klüppelberg, 2016].

Model

- We model the banking network using an inhomogeneous random graph;
- To each possible edge, we assign another random variable E_{ij} which represents the amount of money bank i owes to bank j if edge e_{ij} is present;
- Define each banks initial capital in such a way that will make the network resilient, i.e., making a few banks default will not cause a large portion of banks end the cascade in default;
- Apply shock in the common, which will trigger a cascade (some banks in default, other banks weakened);
- Analyze default cascade process as in [Detering et al.,];

Inhomogeneous Random Graph

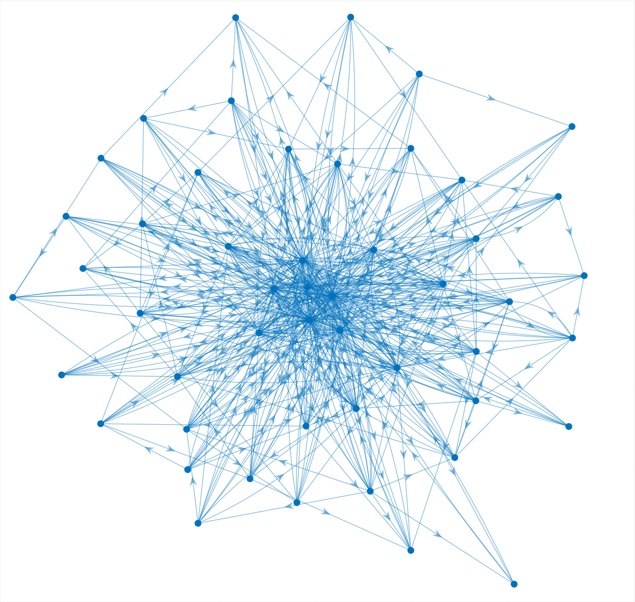
Definition (Inhomogeneus Random Graph Model, [Detering et al., 2019])

Consider a graph with n vertices. To each vertex we assign two deterministic weights $w_i^- = w_i^-(n) \in \mathbb{R}_+$ *and* $w_i^+ = w_i^+(n) \in \mathbb{R}_+$*. The probability that there exists a directed edge from vertex i to vertex j,* $p_{i,j} = p_{i,j}(n)$ *for* $i \neq j$ *is given by:*

$$p_{i,j} = \min\left\{1, \frac{w_i^+ w_i^-}{n}\right\}. \tag{1}$$

Furthermore, we assume that the event that an edge is present happens independent of the presence of all other edges. Call the resulting random graph $G_n(w^-, w^+)$*.*

Example of Inhomogeneous Random Graph



Edge Weights

- To each possible edge in the random graph we assign an edge weight which represent the possible capital exposures between the two banks linked;
- For any pair or vertices (i, j), $i, j \in [n], i \neq j$, there exists a random variable $E_{i,j}$;
- This random variable is defined on a different probability space and the full graph is defined on the product space. On the product space, the edges and exposures are independent;
- We further assume that these exposures are iid and therefore the vector of exposures $(E_{1,j}, E_{2,j}, ..., E_{n,j})$ for bank j is exchangeable.

Threshold function

- To each vertex is assigned a threshold level $c_i(n)$;
- Bank i becomes bankrupt (infected) after a number $c_i(n)$ of its in-neighbors are in default;
- $c_i(n) \in \mathbb{N}_0^\infty$;
- If $c_i(n) = 0$ bank i is in default. If $c_i(n) = \infty$ bank i never goes into default.

Theorem

Theorem ([Detering et al., 2019])

Let $(w^-(n), w^+(n), c(n))_{n\geq 1}$ *be a regular vertex sequence with limiting distribution* $F : \mathbb{R} \times \mathbb{R} \times \mathbb{N}_0^\infty \to [0,1]$. *Let* (W^-, W^+, C) *be a random vector with distribution* F. *Assume* $\mathbb{P}(C = 0) > 0$. *Denote by* $\hat{z}$ *the smallest positive solution of*

$$f(z; (W^-, W^+, C)) := \mathbb{E}[W^+ \mathbb{P}(Poi(W^- z) \geq C)] - z = 0 \tag{2}$$

Let $\mathcal{A}_n$ *denote the final set of infected vertices in* $G_n(w^-, w^+, c)$. *Then:*

1. *For all* $\varepsilon > 0$, $\lim_{n\to\infty} \mathbb{P}(n^{-1}|\mathcal{A}_n| \geq \mathbb{E}[\mathbb{P}(Poi(\hat{z}W^-) \geq C)] - \varepsilon) = 1$
2. *If there exists* $\delta > 0$ *such that the weak derivative* $\mathbb{E}[W^+ W^- \mathbb{P}(Poi(zW^-) = C - 1)\mathbf{1}_{C\geq 1}] - 1$ *is negative for* $z \in (\hat{z} - \delta, \hat{z} + \delta)$, *then*

$$n^{-1}|\mathcal{A}_n| \xrightarrow{p} g(\hat{z}; (W^-, C)) := \mathbb{E}[\mathbb{P}(Poi(\hat{z}W^-) \geq C], \text{as } n \to \infty \tag{3}$$

Theorem - discussion

Our examples will use Theorem 1 to analyze the final fraction of banks in default. We note that:

- Contagion process is analyzed with functional $f(z;(W^-,W^+,C)) = \mathbb{E}[W^+\mathbb{P}(Poi(W^-z) \geq C)] - z$;
- If there is no initial infection is, then $f(0) = 0$;
- The proportion of vertices infected in the end of the cascade process is determined by $g(\hat{z}) = \mathbb{E}[\mathbb{P}(Poi(\hat{z}W^-) \geq C]$;
- Functional g is increasing with $\hat{z}$, the root of (2).

Theorem - Case 1

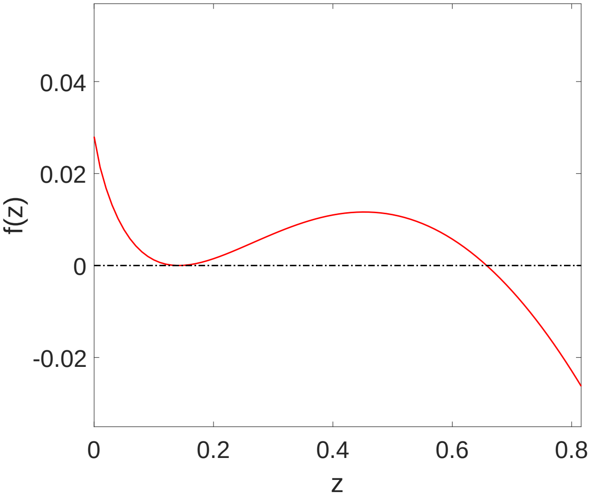


Figure 2: Example to explain case 1 in Theorem 1.

Theorem - Case 2

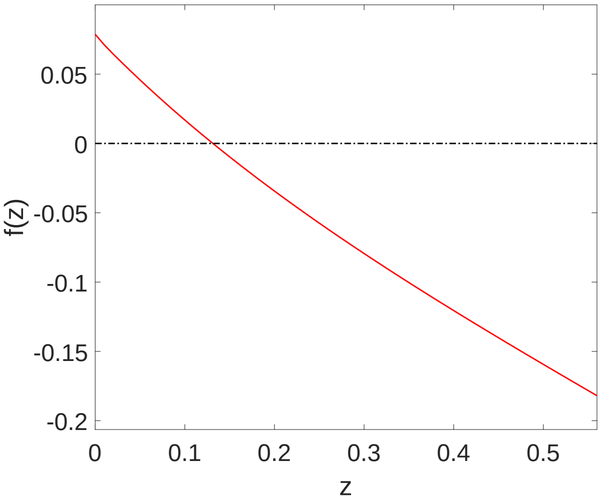


Figure 3: Example to explain case 2 in Theorem 1.

Non-Resilience

- We now define non-resilience for the any system with limiting distribution (W^-, W^+, C) with $\mathbb{P}(C = 0) = 0$;
- In the definition we assume that $M \sim Bernoulli(p)$ and the shocked threshold level is $\tilde{C} = (1 - M)C$.

Definition

A financial system is non-resilient if there exists a constant $\delta > 0$ such that

$$n^{-1}|\mathcal{A}_n| > \delta \text{ with high probability} \tag{4}$$

for any random variable M with parameter $p > 0$.

Non-Resilience

Theorem

Assume that $(w^-(n), w^+(n), c(n))$ *is a regular vertex sequence with limiting distribution* (W^-, W^+, C), $\mathbb{P}(C = 0) = 0$, *and there exists* $z_0 > 0$ *such that*

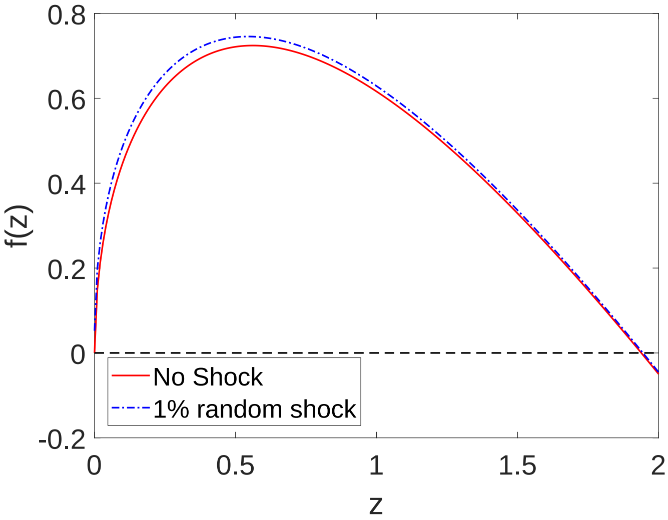
$$f(z) > 0, \text{ for all } 0 < z < z_0. \tag{5}$$

Then for all M with parameter $p > 0$:

$$n^{-1}|\mathcal{A}_n| > g(z_0) \text{ with high probability} \tag{6}$$

independent of p. Therefore, the system is non-resilient.

Non-Resilience Example



Resilience

- Analogously, we now define resilience for the any system with limiting distribution (W^-, W^+, C) with $\mathbb{P}(C = 0) = 0$;
- Again we assume that $M \sim \mathit{Bernoulli}(p)$ and the shocked threshold level is $\tilde{C} = (1 - M)C$.

Definition

A financial system is resilient if for each $\varepsilon > 0$ there exists $\delta > 0$ such that for all $p < \delta$ (i.e., $\mathbb{P}(\tilde{C} = 0) < \delta$)

$$n^{-1}|\mathcal{A}_n| \leq \varepsilon \text{ with high probability} \tag{7}$$

Resilience

Theorem

Assume that $(w^-(n), w^+(n), c(n))$ *is a regular vertex sequence with limiting distribution* (W^-, W^+, C), $\mathbb{P}(C=0)=0$, *and there exists* $z_0 > 0$ *such that*

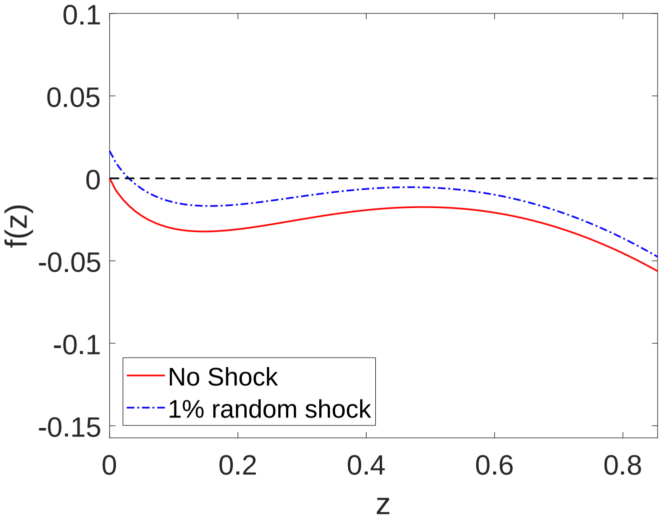
$$\mathbb{E}[W^- W^+ \mathbb{P}(Poi(W^- z) = C-1)\mathbb{1}_{C\geq 1}] - 1 < 0, \ \textit{for all}\ 0 < z < z_0. \tag{8}$$

For every ε, *there exists a* p_0 *such that if* $\mathbb{P}(M=1) < p_0$ *then*

$$n^{-1}|\mathcal{A}_n| < \varepsilon \text{ with high probability.} \tag{9}$$

The system is resilient.

Resilience Example



Hypothetical Threshold Function

- We define the hypothetical threshold function $c_{i,t}(n)$, for $t \geq 0$, as follows:

$$c_{i,t}(n) = \begin{cases} 0, & \text{if } \nu_{i,t} < 0, \\ \inf\{s \in [n] | \sum_{l=1}^{s} E_{il} \geq \nu_{i,t}\}, & \text{if } \nu_{i,t} \geq 0; \end{cases} \tag{10}$$

- The hypothetical threshold level depends on the bank's capital and interbank exposures.

Hypothetical Threshold Function

- The order of the exposures $E_{i,j}$ will result in different threshold level $c_{i,0}$;
- Since all the $E_{i,j}$ are iid random variables, any deterministic permutation $\pi_j : [n-1] \to [n] \setminus \{j\}$ of all the possible exposures of vertex j, $E_{1,j}, E_{2,j}, \ldots, E_{j-1,j}, E_{j+1,j}, \ldots, E_{n,j}$ will have the same multivariate distribution;
- The distribution of the random variable $c_{i,t}$ is the same for all $i \in [n]$;
- [Detering et al., 2019] show that we can use Theorem 1 with the triple (W^-, W^+, C_t) to analyze the default contagion process of the model described by $(W^-, W^+, \nu_t, E_{i,j})$.

Bank's Capital

The capital of bank i will be defined in two different ways:

- *Average capital* (Proposed in [Detering et al., 2019])

$$\nu_{i,0} = \max\{\tau(w_i^-)\mu_i, \max_{j\in[n]\setminus\{i\}} E_{j,i}\}$$

- *Hypothetical capital* (Proposed in this work)

$$\nu_{i,0} = \sum_{j=1}^{\tau_i} E_{i,j}$$

The quantity $\tau_i = \tau(w_i^-)$ is imposed by a regulator.

Shock in the Common Asset

- We want to impose a shock to all banks at once, which would represent a loss in the common assets;
- At time $t > 0$, the capital of bank i given by $\nu_{i,t} = \nu_{i,0} - \alpha_i$;
- $\alpha_i > 0$ conditionally independent (given the value of the asset at time t, α_i is independent of α_j, $i \neq j$);
- $\alpha_i > 0$ is either a constant or another random variable independent of the edges or exposures.

Hypothetical Threshold Function After Shock

- Using the definition of the hypothetical threshold level (10), we can find the pdf of $c_{i,1}$ as a function of the exposures E_{ij} and the bank's capital $\nu_{i,1}$ at time $t = 1$;
- With (W^-, W^+, C_0) and (W^-, W^+, C_1) which describe the system before and after shock, respectively, we calculate functional f in equation (2) and analyze the contagion process.

Contribution to Literature

- Included common shock into default contagion analysis;
- Analytical results for the distribution of $C_{i,t}$ which speeds simulation;
- Adapted resilience and non-resilience theorems to common asset scenario;
- Extend default contagion analysis for continuous time: $(W^{-}, W^{+}, C_{i,t})$. $C_{i,t}$ depends on the value of the common asset at time t;
- Simulation analysis confirms empirical observations in [Cont et al., 2013] and [Elsinger et al., 2006].

Simulation

- We start our simulation by generating a vector of size n with a sample from the weight distribution (W^-, W^+);
- Generate, for each bank i, a vector of exposures $E_{i,j}$;
- Choose $\tau_i = \tau(w_i^-)$ such that the initial network is resilient;
- Calculate the initial capital $\nu_{i,0}$ for each bank i using equations (22) and (22);
- Sample a vector of shocks α_i, and calculate the new capital $\nu_{i,1} = \nu_{i,0} - \alpha_i$;
- Find the threshold level before and after the shock, C_0 and C_1, respectively;
- Calculate functional f before and after shock and compare.

Examples

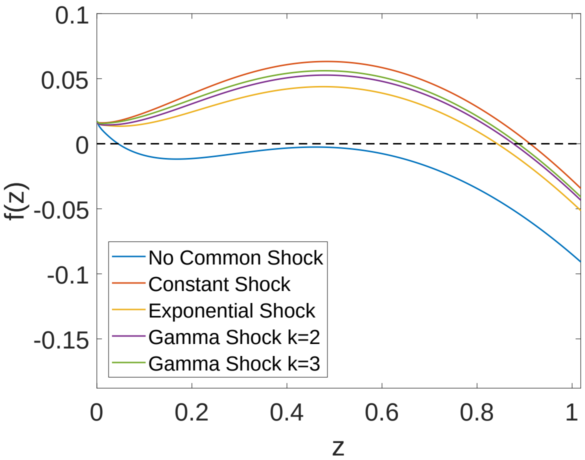
- The examples are split into two parts:
  1. Networks with degree sequence with finite second moment;
  2. Networks with degree sequence with infinite second moment.
- These examples have very distinct behavior, particularly regarding resilience.

Examples with Second Moment

- The vector (W^-, W^+) is sampled from comonotone Pareto random variables with finite second moment;
- In all simulations $n = 50000$;
- Choose $\tau(w_i^-) = 2$. This choice makes the initial network resilient.

Example 1 with Second Moment

Example using *Hypothetical Capital* $\nu_{i,0} = \sum_{j=1}^{\tau_i} E_{i,j}$.



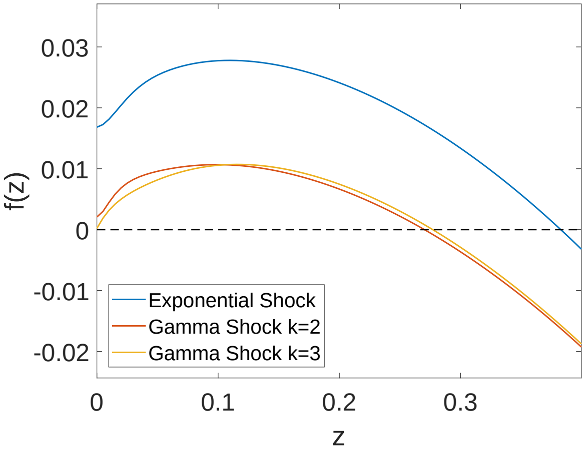
Example 1 with Second Moment

Type of shock	$g(\hat{z})$
No Common Shock	0.0137
Exponential	0.4036
Gamma $k = 2$	0.4261
Gamma $k = 3$	0.4325
Constant	0.4488

Table 1: Proportion of banks in default after cascade process ends.

Example 2 with Second Moment

Example using *Average Capital* $\nu_{i,0} = \max\{\tau(w_i^-)\mu_i, \max_{j\in[n]\setminus\{i\}} E_{j,i}\}$. All shocks have the same expected value.



Example 2 with Second Moment

Type of shock	$g(0)$	$g(\hat{z})$
Exponential	0.0100	0.1618
Gamma $k = 2$	0.0012	0.1083
Gamma $k = 3$	0.00016	0.1076

Table 2: Proportion of banks in default after cascade process ends.

Summary Examples with Second Moment

	Example 1	Example 2	
Type of Shock	$g(\hat{z})$	$g(0)$	$g(\hat{z})$
Exponential	0.4036	0.0100	0.1618
Gamma k=2	0.4261	0.0012	0.1083
Gamma k=3	0.4325	0.00016	0.1076
Constant	0.4488	-	-

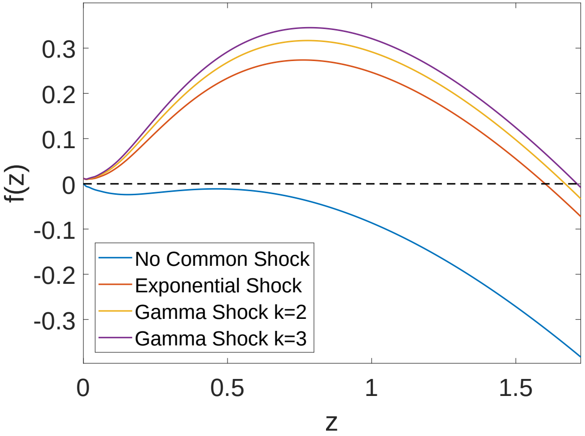
Table 3: Summary. Proportion of banks in default after cascade process ends for each example.

Examples without Second Moment

- The vector (W^-, W^+) is sampled from comonotone Pareto random variables with infinite second moment;
- In all simulations $n = 500000$;
- Choose $\tau(w_i^-) = \max\left\{2, \alpha_c(1+\delta_1)(w_i^-)^{\gamma_c(1+\delta_2)}\right\}$. This choice makes the initial network resilient.

Example 1 without Second Moment

Example using *Average Capital* $\nu_{i,0} = \max\{\tau(w_i^-)\mu_i, \max_{j\in[n]\setminus\{i\}} E_{j,i}\}$.



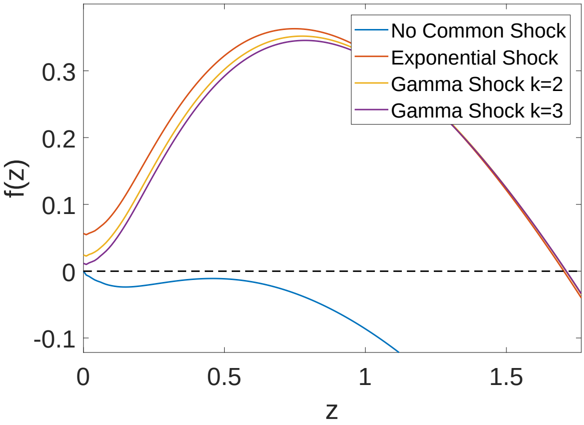
Example 1 without Second Moment

Type of shock	$g(\hat{z})$
Exponential	0.6276
Gamma $k = 2$	0.6740
Gamma $k = 3$	0.7016

Table 4: Proportion of banks in default after cascade process ends.

Example 2 without Second Moment

Example using *Average Capital* $\nu_{i,0} = \max\{\tau(w_i^-)\mu_i, \max_{j\in[n]\setminus\{i\}} E_{j,i}\}$. All shocks have the same expected value.



Example 2 without Second Moment

Type of shock	$g(0)$	$g(\hat{z})$
Exponential	0.0453	0.6963
Gamma $k = 2$	0.0204	0.7016
Gamma $k = 3$	0.0100	0.7017

Table 5: Proportion of banks in default after cascade process ends.

Example 3 without Second Moment

Example using *Average Capital* $\nu_{i,0} = \max\{\tau(w_i^-)\mu_i, \max_{j\in[n]\setminus\{i\}} E_{j,i}\}$. Shock α_i is comonotone with w_i^-.

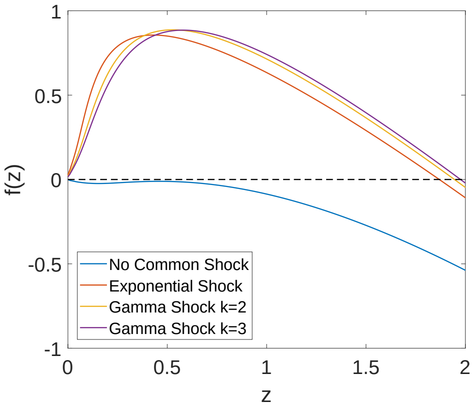


Figure 6: Functional f after different comonotone shocks are applied to initial capital with $\delta_1 = \delta_2 = 0.08$.

Example 3 without Second Moment

Type of shock	$g(\hat{z})$
Exponential	0.7828
Gamma $k = 2$	0.8433
Gamma $k = 3$	0.8704

Table 6: Proportion of banks in default after cascade process ends.

Example 3 without Second Moment

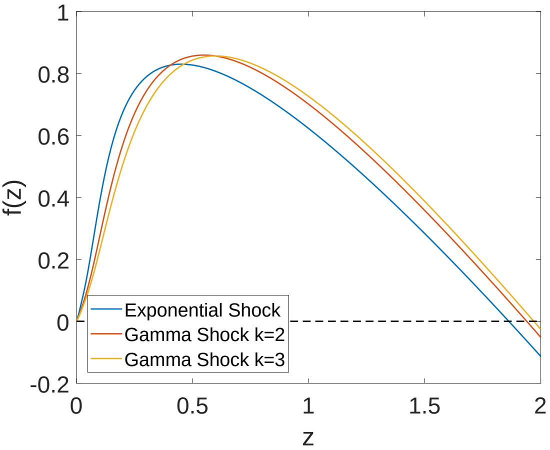


Figure 7: Functional f after different comonotone shocks are applied to initial capital with $\delta_1 = \delta_2 = 0.092$.

Example 3 without Second Moment

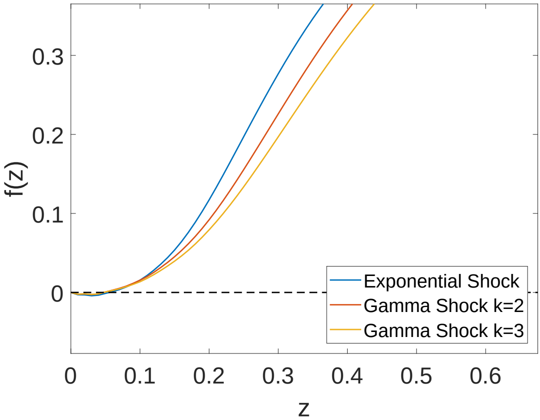


Figure 8: Functional *f* after different comonotone shocks are applied to initial capital with $\delta_1 = 0.2$ and $\delta_2 = 0.3$.

Summary Examples without Second Moment

	Example 1	Example 2		Example 3
Type of Shock	$g(\hat{z})$	$g(0)$	$g(\hat{z})$	$g(\hat{z})$
Exponential	0.6276	0.0453	0.6963	0.7828
Gamma k=2	0.6740	0.0204	0.7016	0.8433
Gamma k=3	0.7016	0.0100	0.7017	0.8704

Table 7: Summary. Proportion of banks in default after cascade process ends for each example.

Conclusion

- We showed that applying a common shock to a resilient network usually makes the resulting network non-resilient;
- In most cases, the common shock causes devastating damage to the system, affecting a large proportion of the network;
- Results presented here, in a probabilistic study, are consistent with empirical studies such as [Elsinger et al., 2006] and [Cont et al., 2013];
- In the comonotone case, it is required a very large shock (low probability, as in [Elsinger et al., 2006]) to trigger a cascade. However, if the contagion process starts, it will affect a very large proportion of the system;
- We showed that the behavior of the system in the absence of a common shock can underestimate the threat to the entire system, as stated in [Cont et al., 2013].

Future Work

- Increasing the capital buffer for each bank can make the system resilient;
- Is it possible to know the value of $\tau(w_i^-)$ for a given loss distribution of the common asset?
- We did not consider bail out. We showed that larger banks have a higher impact to systemic risk. Is there a way (maybe optimal) to bail out specific banks in order to make the system more stable?

Thank you

Thank you!

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